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# SOLUTION OF SOME BOUNDARY PROBLEMS OF HEAT CONDUCTION FOR CYLINDRICAL BODIES PROVIDED WITH LONGITUDINAL FINS

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**Abstract**—The paper deals with a solution of the temperature field in cylindrical bodies consisting of several annuli provided with external or internal longitudinal fins in the shape of annular sectors or prisms. The problem is analysed for boundary conditions varying along the periphery, and for an arbitrary distribution of sources over the cross section. Problems of this type are encountered in thermal-engineering computations of longitudinally finned nuclear reactor fuel elements or of longitudinally finned tubes of heat exchangers. The method evolved in the paper can also be applied to problems concerning laminar flow in non-circular channels.

## NOMENCLATURE

$x, y$, Cartesian coordinates [m], [m];
$r, \varphi$, polar coordinates [m], [rad];
$R$, radius of the interface of the two media [m];
$t$, temperature [°C];
$\lambda$, heat conductivity [kcal/m h °C];
$\alpha$, heat-transfer coefficient [kcal/m$^2$ h °C];
$\rho$, thermal contact resistance [m$^2$h°C/kcal];
$g$, density of heat sources [kcal/m$^3$h].

## 1. INTRODUCTION

ONE OF the interesting problems of heat conduction which is of considerable importance in practical applications, is the analysis of the temperature field in cylindrical bodies with projecting longitudinal fins. Problems of this sort are met with in detailed analyses of heat conduction in finned tubes of heat exchangers and in particular, in longitudinally finned nuclear reactor fuel elements. Figure 1 shows two versions of a fuel element in which this type of problem occurs. The longitudinal fins are considered to be shaped like annular sectors or like prisms. The body is considered as composed of several simple regions and the solutions of heat-conduction equation in individual regions is found in the form of infinite series. Then the solutions in individual regions are bound together by means of boundary conditions on the interfaces between these regions. Thus we receive an infinite system of linear algebraic equations for the integration constants. This system of equations was solved with the help of the method of reduction. The paper does not deal with the analysis of the application of this method in our calculations.

The computation assumes an arbitrary distribution of heat sources across the cross section, and boundary conditions varying along the periphery. Heat conduction in the direction of the $z$-axis is neglected.

It is also assumed in our considerations that the heat conductivity of the material $\lambda$, the

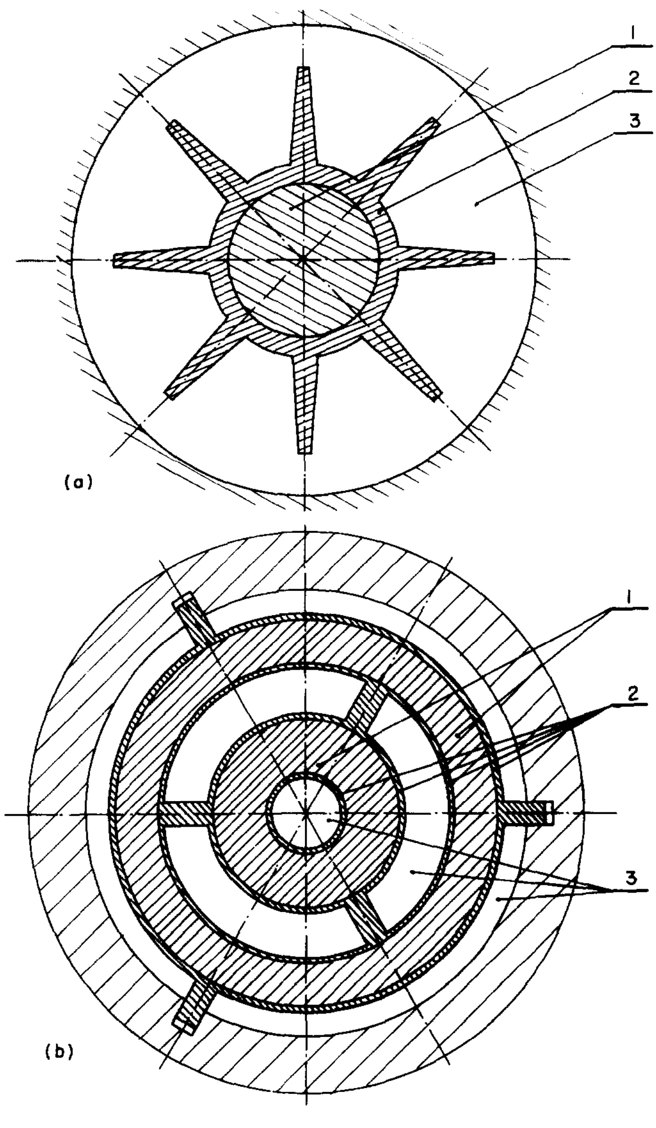


FIG. 1. Some types of longitudinally finned fuel elements in which the examined problem occurs (1—fuel; 2—can; 3—cross section for flow of cooling medium).

heat-transfer coefficient for the boundary condition $\alpha$ and the contact resistance on the interface of the two media $\rho$ are independent of temperature. However, cases including temperature dependent coefficients $\lambda$, $\alpha$, $\rho$ can also be solved on the basis of [2].

## 2. SOLUTION OF HEAT-CONDUCTION EQUATION IN VARIOUS COORDINATE SYSTEMS

### 2.1. *Solution of the heat-conduction equation in polar coordinates*

The heat-conduction equation in the form of

$$-\lambda\,\nabla^2 t = g(\mathbf{r}) \tag{1}$$

may be rewritten for polar coordinates in the form of

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \varphi^2} = -\frac{g(r,\varphi)}{\lambda}. \tag{2}$$

For the sake of simplicity it is assumed that the problem is symmetric with respect to the $x$-axis. Then the solution of the equation (2) may be sought in the form of

$$t = \sum_{i=0}^{\infty} f_i \cos m_i\varphi. \tag{3}$$

First we expand the right-hand side of equation (2) in a Fourier series,

$$\frac{g(r,\varphi)}{\lambda} = \sum_{i=0}^{\infty} g_i(r) \cos m_i\varphi \tag{4}$$

where functions $\cos m_i\varphi$ satisfy conditions (cf. Section 2.3)

$$\int_0^{\phi} \cos m_i\varphi \,.\, \cos m_j\varphi \,\mathrm{d}\varphi = \begin{cases} 0 & i \neq j \\ \dfrac{\phi}{2}\left(1 + \dfrac{\sin 2m_i\phi}{2m_i\phi}\right) & i = j. \end{cases} \tag{5}$$

Multiplying successively equation (4) by functions $\cos m_i\varphi$ and integrating with respect to variable $\varphi$ in the interval of $0 \leqslant \varphi \leqslant \phi$, we obtain, by using (5), the following expression for the Fourier coefficients

$$g_i(r) = \frac{2}{\phi}\,\frac{1}{1 + \dfrac{\sin 2m_i\phi}{2m_i\phi}} \int_0^{\phi} \frac{g(r,\varphi)}{\lambda} \cos m_i\varphi \,\mathrm{d}\varphi. \tag{6}$$

In the case of $g$ = const, we find

$$g_i = 4\frac{g}{\lambda}\,\frac{\sin m_i\phi}{2m_i\phi + \sin 2m_i\phi}. \tag{7}$$

Introducing series (4) and the assumed form of solution (3) in the differential equation (2) we obtain an equation that can be written out for various functions of variable $\varphi$ (i.e. for $\cos m_i\varphi$). The ordinary differential equations thus arrived at are in the form of

$$f_i'' + \frac{1}{r} f_i' - \frac{m_i^2}{r^2} f_i = -g_i(r) \qquad (i = 0, 1, 2, \ldots). \tag{8}$$

Solution of these differential equations can be effected by means of the method of integrating factors [1]. We obtain*

$$f_i = A_i r^{m_i} + B_i r^{-m_i} - r^{m_i} \int_{r_0}^{r} r'^{-(2m_i+1)} \int_{r'_0}^{r'} g_i(r'')\, r''^{m_i+1}\, \mathrm{d}r''\, \mathrm{d}r' \qquad \text{for} \quad m_i \neq 0 \tag{9}$$

$$f_i = A_0 + B_0 \ln r - \int_{r_0}^{r} 1/r \int_{r'_0}^{r'} g_i(r'')\, r''\, \mathrm{d}r'\, \mathrm{d}r' \qquad \text{for} \quad m_i = 0. \tag{10}$$

In the case where none of the values of $m_i$ equals zero, we may, on introducing in series (3), write the solution of equation (2) in the form

$$t(r, \varphi) = \sum_{i=0}^{\infty} [A_i r^{m_i} + B_i r^{-m_i} + Z_i(r)] \cos m_i\varphi; \tag{11}$$

if some of the values of $m_i$ is equal to zero (let that be for $i = 0$), we obtain the solution in the form of

$$t(r, \varphi) = A_0 + B_0 \ln r + z_0(r) + \sum_{i=1}^{\infty} [A_i r^{m_i} + B_i r^{-m_i} + Z_i(r)] \cos m_i\varphi \tag{12}$$

where

$$Z_i(r) = -r^{m_i} \int_{r_0}^{r} r'^{-(2m_i+1)} \int_{r'_0}^{r'} g_i(r'')\, r''^{m_i+1}\, \mathrm{d}r''\, \mathrm{d}r'. \tag{13}$$

*Note* 1. In the case of the region being circular, constants $B_i$ are equal to zero for physical reasons.

*Note* 2. As Section 2.3 will indicate, it is profitable to modify solution (11) by the addition of a constant (let us denote it by $t_0$). The solution thus obtained in the form of

$$t(r, \varphi) = t_0 + \sum_{i=0}^{\infty} [A_i r^{m_i} + B_i r^{-m_i} + Z_i(r)] \cos m_i\varphi \tag{11a}$$

also satisfies the differential equation (2).

### 2.2. *Solution of the heat-conduction equation in the Cartesian coordinates*

Assuming again that the problem is symmetrical with respect to the $x$-axis, we can write the heat-conduction equation as

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = -\frac{g(x, y)}{\lambda} \tag{14}$$

and seak its solution in the form of

$$t = \sum_{i=0}^{\infty} f_i \cos m_i y. \tag{15}$$

Using a procedure analogous to that of the preceding section, we obtain for the solution of equation (14) a series (assuming none of the values $m_i$ equals zero)

$$t(x, y) = t_0 + \sum_{i=0}^{\infty} [A_i\, \mathrm{e}^{m_i x} + B_i\, \mathrm{e}^{-m_i x} + Z_i(x)] \cos m_i y \tag{16}$$

* The integrals must be thought of as indefinite integrals: $r_0$ and $r'_0$ are the suitably chosen limlts.

where

$$Z_i(x) = -e^{m_i x}\int_{x_0}^{x} e^{-2m_i x'}\int_{x_0'}^{x'} g_i(x'')\, e^{m_i x''}\, dx''\, dx'. \tag{17}$$

The Fourier coefficients of function $g(x, y)$ can be determined in a manner similar to that used in Section 2.1. Thus

$$g_i(x) = \frac{2}{Y}\frac{1}{1 + \dfrac{\sin 2m_i Y}{2m_i Y}}\int_0^Y \frac{g(x, y)}{\lambda}\cos m_i y\, dy \tag{18}$$

or for $g$ = constant

$$g_i = 4\frac{g}{\lambda}\frac{\sin m_i Y}{2m_i Y + \sin 2m_i Y}. \tag{19}$$

2.3. *Determining the values $m_i$*

The boundary conditions on the surface of the regions examined and at the interface of the two media will be dealt with in detail in Section 3. In what follows we shall mention the boundary conditions only in so far as they are required for the determination of the values of $m_i$.

2.3.1. *The examined region is circular or annular.* In the case where the examined region is circular or annular, we obtain for reasons of cyclicity of the solution (in dependence of angle $\varphi$) the following values of $m_i$:

$$m_i = 0, 1, 2, \ldots. \tag{20}$$

If, however, the solution is to be repeated $N$-times along the periphery [see e.g. Fig. 1(a) where $N = 8$], we obtain

$$m_i = 0, N, 2N, \ldots. \tag{21}$$

2.3.2. *The examined region is in the form of an annular sector.* Let us assume that along the radius vector for $\varphi = \phi$ (cf. Fig. 2) the third boundary condition applies in the form of

$$-\frac{1}{r}\lambda\left.\frac{\partial t(r, \varphi)}{\partial\varphi}\right|_{\varphi=\phi} = \alpha(r)\left[t(r, \phi) - t_0\right] \tag{22}$$

where $t_0$ is the ambient temperature. Introducing form (11a) to condition (22) we obtain

$$\lambda\, 1/r \sum\left[A_i r^{m_i} + B_i r^{-m_i} + Z_i(r)\right] m_i \sin m_i\phi = \alpha(r)\sum\left[A_i r^{m_i} + B_i r^{-m_i} + Z_i(r)\right]\cos m_i\phi. \tag{23}$$

If the boundary condition is required to be satisfied for the various harmonic components, we obtain for the values of $m_i$ a transcendental equation in the form

$$m\phi \tan m\phi = \frac{\alpha(r)}{\lambda} r\phi.$$

As this equation implies, product $\alpha(r) . r$ must be constant, i.e. the course of the heat-transfer coefficient along radius-vector for $\varphi = \phi$ must be in the form of

$$\alpha(r) = \text{const}/r.$$

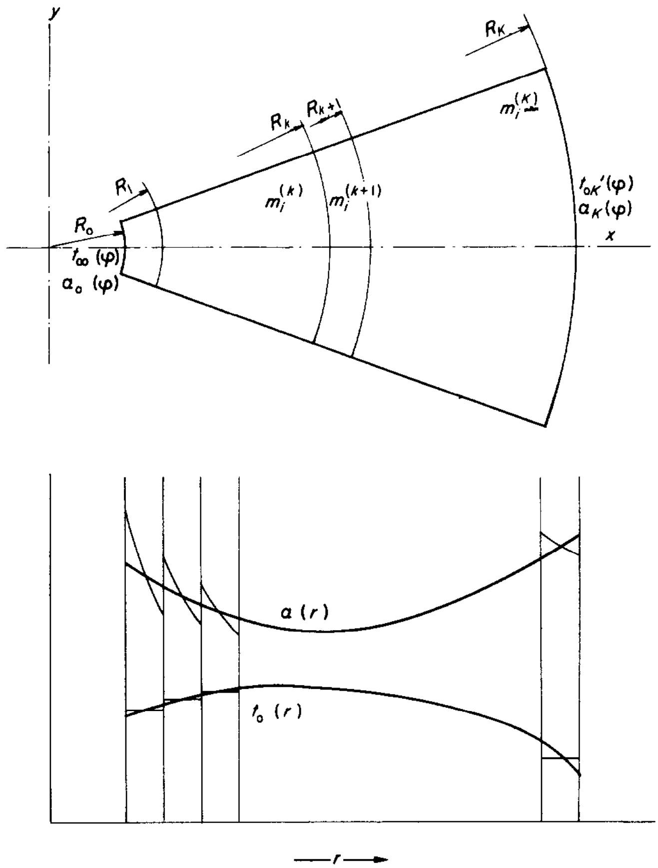


FIG. 2. Sketch of fin in the shape of annular sector with third boundary condition.

Thus we obtain the following equation for the determination of the values of $m$

$$\chi \tan \chi = C \tag{24}$$

where

$$\chi = m\phi, \qquad C = \frac{\alpha(r_0)\, r_0 \phi}{\lambda} \tag{25}$$

and $r_0$ is a suitably chosen radius.

It can be demonstrated that the pair of roots of equation (24) satisfy conditions (5). In the case $\alpha \to 0$ we obtain the so called second boundary condition. Numbers $\chi_i$ assume the values of

$$\chi_i = 0, \pi, 2\pi, \ldots$$

whence

$$m_i = 0, \frac{\pi}{\phi}, \frac{2\pi}{\phi}, \ldots.$$

On the other hand, for $\alpha \to \infty$ we obtain the so called first boundary condition. The values of $m_i$ are

$$m_i = \frac{1}{2}\frac{\pi}{\phi}, \frac{3}{2}\frac{\pi}{\phi}, \ldots.$$

2.3.3. *The examined region is prismatic.* We assume that the third boundary condition

$$-\lambda \frac{\partial t(x, y)}{\partial y}\bigg|_{y=Y} = \alpha(x)\,[t(x, Y) - t_0] \tag{26}$$

now applies along the straight line $y = Y$. Analogously to Section 2.3.2 this implies that the values of $m_i$ are the roots of the equation

$$mY \tan mY = \frac{\alpha(x)}{\lambda} Y.$$

It is, therefore, indicated that along the line $y = Y$ we must have $\alpha(x) = \text{const.}$

2.4. *Resultant symbolic representation of expressions for variations of temperature t and derivative* $\partial t/\partial r$

2.4.1. *Circular region.* In view of Section 2.3, series (12) for the region $k$ may be written in matrix symbolics as follows:

$$t_k(r, \varphi) = {}^k\mathbf{S}_\varphi({}^k_1\mathbf{M}_r\, {}_1\mathbf{C}_k + {}^k\mathbf{Z}_r) \tag{27}$$

where the row matrix ${}^k\mathbf{S}_\varphi$, square matrix ${}^k_1\mathbf{M}_r$ and vectors ${}_1\mathbf{C}_k$ and ${}^k\mathbf{Z}_r$ are respectively in the form of

$${}^k\mathbf{S}_\varphi = [\cos m_0^{(k)}\varphi, \cos m_1^{(k)}\varphi, \cos m_2^{(k)}\varphi, \ldots]$$

$${}^k_1\mathbf{M}_r = \begin{bmatrix} 1 & & \\ & r^{m_1(k)} & \\ & & r^{m_2(k)} \\ & & & \ddots \end{bmatrix}, \quad {}_1\mathbf{C}_k = \begin{bmatrix} A_0^{(k)} \\ A_1^{(k)} \\ A_2^{(k)} \\ \vdots \end{bmatrix}, \quad {}^k\mathbf{Z}_r = \begin{bmatrix} Z_0(r) \\ Z_1(r) \\ Z_2(r) \\ \vdots \end{bmatrix}$$

and where functions $Z_i(r)$ are given by expression (13). On carrying out differentiation $\partial t_k/\partial r$ we obtain the expression

$$\frac{\partial t_k(r, \varphi)}{\partial r} = {}^k\mathbf{S}_\varphi({}^k_1\mathbf{M}'_r\, {}_1\mathbf{C}_k + {}^k\mathbf{Z}'_r) \tag{28}$$

where matrices ${}^k_1\mathbf{M}'_r$ and ${}^k\mathbf{Z}'_r$ are derivatives of matrices ${}^k_1\mathbf{M}_r$ and ${}^k\mathbf{Z}_r$.

2.4.2. *Annular region.* In the case of an annular region we obtain analogously to Section 2.4.1

$$t_k(r, \varphi) = {}^k\mathbf{S}_\varphi({}^k_2\mathbf{M}_r\, C_k + {}^k\mathbf{Z}_r) \tag{29}$$

where the rectangular matrix ${}^k_2\mathbf{M}_r$ and vector $\mathbf{C}_k$ are respectively in the form of

$$ {}^k_2\mathbf{M}_r = \begin{bmatrix} 1, & \ln r & & & & & \\ & & r^{m_1^{(k)}}, & r^{-m_1^{(k)}} & & & \\ & & & & r^{m_2^{(k)}}, & r^{-m_2^{(k)}} & \\ & & & & & & \ddots \end{bmatrix}, \quad \mathbf{C}_k = \begin{bmatrix} A_0^{(k)} \\ B_0^{(k)} \\ A_1^{(k)} \\ B_1^{(k)} \\ \vdots \end{bmatrix}. $$

On carrying out differentiation $\partial t_k/\partial r$ we obtain

$$ \frac{\partial t_k(r, \varphi)}{\partial r} = {}^k\mathbf{S}_\varphi({}^k_2\mathbf{M}'_r\,\mathbf{C}_k + {}^k\mathbf{Z}'_r) \tag{30} $$

where ${}^k_2\mathbf{M}'_r$ is the derivative of matrix ${}^k_2\mathbf{M}_r$.

2.4.3. *Annular sector.* If the third boundary condition is applied along the radius vector for $\varphi = \phi$, the temperature distribution is given by the expression [cf. equation (11a)]:

$$ t_k(r, \varphi) = t_0 + {}^k\mathbf{S}_\varphi({}^k_3\mathbf{M}_r\,\mathbf{C}_k + {}^k\mathbf{Z}_r) \tag{31} $$

where

$$ {}^k_3\mathbf{M}_r = \begin{bmatrix} r^{m_0^{(k)}}, & r^{-m_0^{(k)}} & & \\ & & r^{m_1^{(k)}}, & r^{-m_1^{(k)}} \\ & & & \ddots \end{bmatrix}. $$

The meaning of the other matrices and vectors is identical with that of the preceding sections; the values of $m_i^{(k)}$ are determined by solving the equation (24).

From the point of view of the solution of the boundary problems, it is convenient to expand the constant $t_0$ in a Fourier series with respect to functions $\cos m_i^{(k)}\varphi$. Relation (31) can be then rewritten in the form of

$$ t_k(r, \varphi) = {}^k\mathbf{S}_\varphi({}^k_3\mathbf{M}_r\,\mathbf{C}_k + {}^k\mathbf{Z}_r + {}^k\mathbf{T}_0) \tag{31a} $$

where

$$ {}^k\mathbf{T}_0 = 4\,{}^kt_0 \begin{bmatrix} \dfrac{\sin m_0^{(k)}\phi}{2m_0^{(k)}\phi + \sin 2m_0^{(k)}\phi} \\ \dfrac{\sin m_1^{(k)}\phi}{2m_1^{(k)}\phi + \sin 2m_1^{(k)}\phi} \\ \vdots \end{bmatrix} $$

On carrying out differentiation $\partial t_k/\partial r$ we obtain the expression

$$ \frac{\partial t_k(r, \varphi)}{\partial r} = {}^k\mathbf{S}_\varphi({}^k_3\mathbf{M}'_r\,\mathbf{C}_k + {}^k\mathbf{Z}'_r). \tag{32} $$

2.4.4. *Prismatic region.* In view of the expression (16) and Section 2.3.3, the relation for the distribution of temperature $t_k(x, y)$ may be written as follows:

$$ t_k(x, y) = t_0 + {}^k\mathbf{S}_y({}^k\mathbf{M}_x\,\mathbf{C}_k + {}^k\mathbf{Z}_x) \tag{33} $$

where

$$
{}^k\mathbf{S}_y = [\cos m_0^{(k)}y, \cos m_1^{(k)}y, \ldots]
$$

$$
{}^k\mathbf{M}_x = \begin{bmatrix} e^{m_0^{(k)}x}, & e^{-m_0^{(k)}x} & & \\ & & e^{m_1^{(k)}x}, & e^{-m_1^{(k)}x} \\ & & & \ddots \end{bmatrix}, \quad {}^k\mathbf{Z}_x = \begin{bmatrix} Z_0(x) \\ Z_1(x) \\ \vdots \end{bmatrix}
$$

and functions $Z_i(x)$ are given by expression (17). A modification similar to that of Section 2.4.3 will give

$$
t_k(x, y) = {}^k\mathbf{S}_y({}^k\mathbf{M}_x \mathbf{C}_k + {}^k\mathbf{Z}_x + {}^k\mathbf{T}_0) \tag{33a}
$$

where vector ${}^k\mathbf{T}_0$ is of the same form as in Section 2.4.3 with the only exception that angle $\phi$ has been replaced by the value $Y$.

The derivative $\partial t_k/\partial x$ is in the form

$$
\frac{\partial t_k(x, y)}{\partial x} = {}^k\mathbf{S}_y({}^k\mathbf{M}'_x \mathbf{C}_k + {}^k\mathbf{Z}'_x). \tag{34}
$$

2.5. *Matrix notation of the product of two Fourier series*

A product of two Fourier series in the form of

$$
\mathbf{SU}.\mathbf{SV}
$$

where the row matrix $\mathbf{S}$ is of the same form as in Section 2.4 and vectors $\mathbf{U}$ and $\mathbf{V}$ contain Fourier coefficients $u_i$ or $v_i$ can be written as follows:

$$
\mathbf{SK}_U\mathbf{V}. \tag{35}
$$

Matrix $\mathbf{K}_U$ contains Fourier coefficients $u_i$ and is in the form of

$$
\mathbf{K}_U = \tfrac{1}{2}\begin{bmatrix} 2u_0, & u_1 & u_2 & u_3 & \ldots \\ 2u_1, & 2u_0 + u_2, & u_1 + u_3, & u_2 + u_4, & \ldots \\ 2u_2, & u_1 + u_3, & 2u_0 + u_4, & u_1 + u_5, & \ldots \\ 2u_3, & u_2 + u_4, & u_1 + u_5, & 2u_0 + u_6, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} \tag{36}
$$

## 3. BOUNDARY CONDITIONS

This chapter will deal with various types of boundary conditions on the surface of a body and at the interfaces of different regions; we shall set up systems of equations which follow from these boundary conditions for integration constants $\mathbf{C}$. The solution of the boundary problems, i.e. the grouping of equations into the resultant system of equations describing the whole boundary problem, will be investigated in the next chapter.

3.1. *The third boundary condition on the surface of an annulus*

We shall write the third boundary condition on the surface of an annulus for both external and the internal surfaces of the annular region (cf. also [2])

$$\mp \lambda \frac{\partial t(r,\varphi)}{\partial r}\bigg|_{r=R} = \alpha(\varphi)\left[t(R,\varphi) - t_0(\varphi)\right] \tag{37}$$

where $\alpha(\varphi)$ and $t_0(\varphi)$ are the given functions. The upper sign in condition (37) refers to the case of radius $R$ being the outer radius of the annulus, the lower to the inner radius.

We substitute now for $t(r, \varphi)$ and $\partial t(r, \varphi)/\partial r$ from (29) and (30) to (37) and expand $\alpha(\varphi)$ and $t_0(\varphi)$ in a Fourier series. By means of relation (35) for a product of two Fourier series we obtain the resultant equation to be written out in a system of equations for the various functions of argument $\varphi$. We get

$$(\mathbf{K}_\alpha\, {}_2\mathbf{M}_R \pm \lambda\, {}_2\mathbf{M}'_R)\,\mathbf{C} = \mp\, \mathbf{Z}'_R + \mathbf{K}_\alpha(\mathbf{T}_{0R} - \mathbf{Z}_R) \tag{38}$$

where vector $\mathbf{T}_{0R}$ contains the Fourier coefficients of function $t_0(\varphi)$. In the case of $\alpha(\varphi) = \alpha = \text{const.}$, matrix $\mathbf{K}_\alpha$ reduces to matrix $\alpha\mathbf{I}$ where $\mathbf{I}$ is the unit matrix.

The same result would be obtained if we introduce into the equation (37) the derived Fourier series for $t(r, \varphi)$, $\partial t/\partial r$, $\alpha(\varphi)$ and $t_0(\varphi)$, multiply the equation thus obtained successively by functions

$$\frac{1}{\pi} \qquad \text{for} \quad m_i = 0$$

$$\frac{2}{\pi}\cos m_i\varphi \qquad \text{for} \quad m_i \neq 0$$

and integrate it within the interval of $0 \leqslant \varphi \leqslant \pi$.

### 3.2. *Boundary condition at the interface of two annuli or at the interface of an annulus and a circle*

At the interface of two regions of radius $R$ the following boundary conditions apply

$$t_k(R,\varphi) = t_{k+1}(R,\varphi) - \lambda_{k+1}\,\rho(\varphi)\frac{\partial t_{k+1}}{\partial r}\bigg|_{r=R} \tag{39}$$

$$\lambda_k \frac{\partial t_k}{\partial r}\bigg|_{r=R} = \lambda_{k+1}\frac{\partial t_{k+1}}{\partial r}\bigg|_{r=R} \tag{40}$$

where $\rho(\varphi)$ is the given function of the thermal contact resistance. Introducing in conditions (39), (40), the series for $t(r, \varphi)$ and $\partial t/\partial r$ and the Fourier series for contact resistance $\rho(\varphi)$, we obtain equations that can be written out for various functions $\cos m_i\varphi$ (as Section 2.3.1 implies $m_i^{(k)} = m_i^{(k+1)}$ for $i = 0, 1, 2, \ldots$). We arrive at a system of equations which we write in matrix notation as follows:

$$ {}^k_2\mathbf{M}_R\,\mathbf{C}_k + {}^k\mathbf{Z}_R = ({}^{k+1}_{\ \ 2}\mathbf{M}_R - \lambda_{k+1}\,\mathbf{K}\rho\;{}^{k+1}_{\ \ 2}\mathbf{M}'_R)\,\mathbf{C}_{k+1} + {}^{k+1}\mathbf{Z}_R - \lambda_{k+1}\,\mathbf{K}\rho\;{}^{k+1}\mathbf{Z}'_R \tag{41}$$

$$\lambda_k({}^k_2\mathbf{M}'_R\,\mathbf{C}_k + {}^k\mathbf{Z}'_R) = \lambda_{k+1}({}^{k+1}_{\ \ 2}\mathbf{M}'_R\,\mathbf{C}_{k+1} + {}^{k+1}\mathbf{Z}'_R). \tag{42}$$

In the case when region $k$ is circular, matrices ${}^k_1\mathbf{M}_R$, ${}^k_1\mathbf{M}'_R$ and vector ${}_1\mathbf{C}_k$ are used instead of ${}^k_2\mathbf{M}_R$, ${}^k_2\mathbf{M}'_R$ and $\mathbf{C}_k$ on the left-hand side of equations (41) and (42).

### 3.3. *The third boundary condition on the surface of an annular sector for $\varphi = \phi$*

Let us assume that both the heat-transfer coefficient $\alpha(r)$ and the ambient temperature $t_0(r)$ vary arbitrarily along the boundary of the annular sector for $\varphi = \phi$ except for the limitation that the boundary condition remains symmetric with respect to the $x$-axis. As shown by the results of Section 2.3.2 it was necessary to solve the problem by the following approximation: we divided the whole region into several sub-regions by means of circles with radii $r = R_k$. The given function for the heat-transfer coefficient $\alpha(r)$ on the surface of the region was then approximated by a set of "steps" of the type const/$r$ on the surfaces of individual sub-regions. The mean ambient temperature ${}^k t_0$ for a sub-region is determined as the mean value of the function $t_0(r)$ in the interval $R_{k-1} \leqslant r < R_k$.

Along the circle of radius $R_k$ within the interval of $0 \leqslant \varphi \leqslant \phi$ the condition of equality of temperatures and heat flow must obviously hold

$$t_k(R_k, \varphi) = t_{k+1}(R_k, \varphi) \tag{43}$$

$$\left.\frac{\partial t_k(R_k, \varphi)}{\partial r}\right|_{r=R_k} = \left.\frac{\partial t_{k+1}(R_k, \varphi)}{\partial r}\right|_{r=R_k}. \tag{44}$$

At variance to Section 3.2 the boundary conditions (43) and (44) cannot be fulfilled in this case for each harmonic component of the series because the values of $m_i^{(k)}$ and $m_i^{(k+1)}$ differ from each other. Therefore we fulfil the conditions only approximately for a finite number of expansion terms.

We introduce in condition (43) the series for $t(r, \varphi)$ in the form of (31a), multiply this equation successively by functions

$$\frac{2}{\phi} \frac{1}{1 + \dfrac{\sin 2m_i^{(k+1)}\phi}{2m_i^{(k+1)}\phi}} \cos m_i^{(k+1)}\varphi \tag{45}$$

and integrate it within the interval of $0 \leqslant \varphi \leqslant \phi$. The system of equations arrived at through this procedure can, after rearrangement, be expressed summarily in matrix notation as follows

$$\boldsymbol{\phi}^{(k+1,k)}({}^k_3\mathbf{M}_{R_k}\mathbf{C}_k + {}^k\mathbf{Z}_{R_k} + {}^k\mathbf{T}_0) = {}^{k+1}_{\ \ 3}\mathbf{M}_{R_k}\mathbf{C}_{k+1} + {}^{k+1}\mathbf{Z}_{R_k} + {}^{k+1}\mathbf{T}_0. \tag{46}$$

A similar procedure adopted for boundary condition (44) differs in that the functions of the form of (45) contain the values of $m_i^{(k)}$. We deduce that

$${}^k_3\mathbf{M}'_{R_k}\mathbf{C}_k + {}^k\mathbf{Z}'_{R_k} = \boldsymbol{\phi}^{(k,k+1)}({}^{k+1}_{\ \ 3}\mathbf{M}'_{R_k}\mathbf{C}_{k+1} + {}^{k+1}\mathbf{Z}'_{R_k}). \tag{47}$$

The elements of matrix $\boldsymbol{\phi}^{(k+1,k)}$ (or $\boldsymbol{\phi}^{(k,k+1)}$) are

$$\phi_{i,j}^{(k+1,k)} = \frac{2}{\phi} \frac{1}{1 + \dfrac{\sin 2m_i^{(k+1)}\phi}{2m_i^{(k+1)}\phi}} \int_0^{\phi} \cos m_i^{(k+1)}\varphi \,.\, \cos m_j^{(k)}\varphi \, \mathrm{d}\varphi$$

and after integration

$$\phi_{i,j}^{(k+1,k)} = \frac{1}{1 + \dfrac{\sin 2m_i^{(k+1)}\phi}{2m_i^{(k+1)}\phi}} \left[\frac{\sin(m_i^{(k+1)} + m_j^{(k)})\,\phi}{(m_i^{(k+1)} + m_j^{(k)})\,\phi} + \frac{\sin(m_i^{(k+1)} - m_j^{(k)})\,\phi}{(m_i^{(k+1)} - m_j^{(k)})\,\phi}\right]. \tag{48}$$

For a prismatic fin we can proceed in an analogous manner.

### 3.4. *The third boundary condition on the surface of an annular sector for $r = R_k$ and $r = R_0$*

Let us now deal with the third boundary condition for radius $R_k$ (or $R_0$) which is in the form of

$$\mp \lambda \left.\frac{\partial t}{\partial r}\right|_{r=R} = \alpha(\varphi)\,[t(R, \varphi) - t_0(\varphi)] \tag{49}$$

(the upper sign applies to radius $R_k$, the lower to radius $R_0$). By multiplying this condition successively by functions in the form of (45) and integrating it within the interval of $0 \leqslant \varphi \leqslant \phi$ we obtain a system of equations that can be summarily written as follows

$$({}_1\mathbf{X}\,{}_3\mathbf{M}_R \pm \lambda\,{}_3\mathbf{M}'_R)\,\mathbf{C} = \mp\,\lambda\,\mathbf{Z}'_R + {}_2\mathbf{X} - {}_1\mathbf{X}(\mathbf{Z}_R - \mathbf{T}_0) \tag{50}$$

where the elements of matrix ${}_1\mathbf{X}$ and vector ${}_2\mathbf{X}$ are in the form of

$${}_1x_{i,j} = \xi \int_0^{\phi} \alpha(\varphi) \cos m_i\varphi \,.\, \cos m_j\varphi \, \mathrm{d}\varphi$$

$${}_2x_i = \xi \int_0^{\phi} \alpha(\varphi)\, t_0(\varphi) \cos m_i\varphi \, \mathrm{d}\varphi$$

and where

$$\xi = \frac{2}{\phi} \frac{1}{1 + \dfrac{\sin 2m_i\phi}{2m_i\phi}}.$$

In the case when for radius $R_k$ (or $R_0$) we have $\alpha_k = \text{const.}$, $t_{0R_k} = \text{const.}$ (or $\alpha_0 = \text{const.}$, $t_{0R_k} = \text{const.}$) we obtain the following simple relations

$$\left.\begin{aligned} (\alpha_k\,{}_3^k\mathbf{M}_{Rk} + \lambda\,{}_3^k\mathbf{M}'_{R_k})\,\mathbf{C}_k &= -\,\lambda\,{}^k\mathbf{Z}'_{R_k} - \alpha_k({}^k\mathbf{Z}_{Rk} + {}^k\mathbf{T}_0 - \mathbf{T}_{0R_k}) \\ (\alpha_0\,{}_3^1\mathbf{M}_{R_0} - \lambda\,{}_3^1\mathbf{M}'_{R_0})\,\mathbf{C}_1 &= \lambda\,{}^1\mathbf{Z}'_{R_0} - \alpha_0({}^1\mathbf{Z}_{R_0} + \mathbf{T}_0 - \mathbf{T}_{0R_0}). \end{aligned}\right\} \tag{51}$$

For a prismatic fin we can proceed in an analogous manner.

### 3.5. *Boundary conditions at the interface between annular region and annular sector in a "concentric" arrangement*

Assuming that on radius $R$ (Fig. 3) within the interval of $\phi < \varphi \leqslant \phi_1$ the boundary condition of

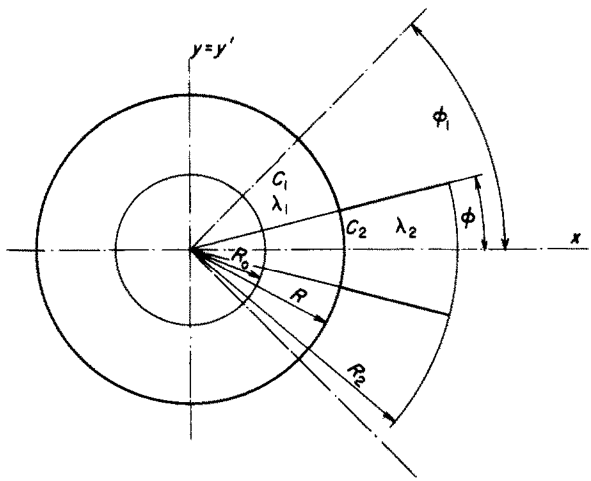


FIG. 3. Annular sector attaching to annular region in concentric arrangement.

the third type exists and the coefficients of thermal conductivity there are $\lambda_1$ and $\lambda_2$, we can write the boundary conditions as follows

$$\lambda_1 \left.\frac{\partial t_1}{\partial r}\right|_{r=R} = -\alpha(\varphi)\left[t_1(R,\varphi) - t_0(\varphi)\right] \qquad \text{for} \quad \phi < \varphi \leqslant \phi_1 \tag{52}$$

$$\lambda_1 \left.\frac{\partial t_1}{\partial r}\right|_{r=R} = \lambda_2 \left.\frac{\partial t_2}{\partial r}\right|_{r=R} \qquad \text{for} \quad 0 \leqslant \varphi \leqslant \phi \tag{53}$$

$$t_1(R,\varphi) = t_2(R,\varphi) \qquad \text{for} \quad 0 \leqslant \varphi \leqslant \phi. \tag{54}$$

Difficulties encountered in the solution of a boundary problem of this sort are due to the fact that two different types of boundary conditions apply to a part of the annulus within the interval of $0 \leqslant \varphi \leqslant \phi_1$. To overcome that, we shall extend the course of $\alpha(\varphi)$ hitherto defined in the interval of $\phi < \varphi \leqslant \phi_1$ only, to the interval of $0 \leqslant \varphi \leqslant \phi$, and put there $\alpha(\varphi) = 0$. The boundary condition for radius $R$ can then be written in the following form

$$\lambda_1 \left.\frac{\partial t_1}{\partial r}\right|_{r=R} = -\alpha(\varphi)\left[t_1(R,\varphi) - t_0(\varphi)\right] + \delta(\varphi)\lambda_2 \left.\frac{\partial t_2}{\partial r}\right|_{r=R} \qquad \text{for} \quad 0 \leqslant \varphi \leqslant \phi_1. \tag{55}$$

This equation comprises both boundary conditions (52) and (53). Function $\delta(\varphi)$ takes the form of

$$\delta(\varphi) = \begin{cases} 1 & \text{for} \quad 0 \leqslant \varphi \leqslant \phi \\ 0 & \text{for} \quad \phi < \varphi \leqslant \phi_1. \end{cases}$$

By multiplying the equation (55) successively by functions

$$\frac{1}{\phi_1} \qquad \text{for} \quad m_i^{(1)} = 0$$

$$\frac{2}{\phi_1}\cos m_i^{(1)}\varphi \qquad \text{for} \quad m_i^{(1)} \neq 0$$

and integrating it within the interval of $0 \leqslant \varphi \leqslant \phi_1$ we get a system of equations which may be summarily written as follows

$$\lambda_1({}^1_2\mathbf{M}'_R\mathbf{C}_1 + {}^1\mathbf{Z}'_R) = -\mathbf{K}_\alpha[{}^1_2\mathbf{M}_R\,\mathbf{C}_1 + {}^1\mathbf{Z}_R + \mathbf{T}_{0R}] + \lambda_2\boldsymbol{\Theta}[{}^2_3\mathbf{M}'_R\,\mathbf{C}_2 + {}^2\mathbf{Z}'_R] \tag{56}$$

where vector $\mathbf{T}_{0R}$ contains the Fourier coefficients of function $t_0(\varphi)$. The elements of matrix $\boldsymbol{\Theta}$ are in the form of

$$\Theta_{i,j} = \frac{\varepsilon}{\phi_1}\int_0^{\phi} \cos m_j^{(2)}\varphi \,.\, \cos m_i^{(1)}\varphi \,\mathrm{d}\varphi \tag{57}$$

where

$$\varepsilon = \begin{cases} 1 & \text{for} \quad m_i^{(1)} = 0 \\ 2 & \text{for} \quad m_i^{(2)} \neq 0. \end{cases}$$

Integrating we obtain

$$\Theta_{i,j} = \varepsilon'\left[\frac{\sin(m_i^{(1)} + m_j^{(2)})\,\phi}{(m_i^{(1)} + m_j^{(2)})\,\phi_1} + \frac{\sin(m_i^{(1)} - m_j^{(2)})\,\phi}{(m_i^{(1)} - m_j^{(2)})\,\phi_1}\right]$$

where

$$\varepsilon' = \begin{cases} \frac{1}{2} & \text{for} \quad m_i^{(1)} = 0 \\ 1 & \text{for} \quad m_i^{(1)} \neq 0. \end{cases}$$

*Note.* Because the integrand equals zero $[\delta(\varphi) = 0]$ within the interval of $\phi < \varphi \leqslant \phi_1$, the integration of integral (57) is carried out in the interval of $0 \leqslant \varphi \leqslant \phi$ only.

With boundary condition (54) we shall proceed in a manner analogous to that used in Section 3.3 for condition (43). Multiplying the equation (54) by functions of the form of (45) using the values of $m_i^{(2)}$, we get, after rearrangement

$$\tfrac{2}{3}\mathbf{M}_R\,\mathbf{C}_2 + {}^2\mathbf{Z}_R + {}^2\mathbf{T}_0 = \boldsymbol{\psi}[\tfrac{1}{2}\mathbf{M}_R\,\mathbf{C}_1 + {}^1\mathbf{Z}_R] \tag{58}$$

where matrix $\boldsymbol{\psi}$ has elements in the form of

$$\Psi_{i,j} = \frac{2}{\phi}\,\frac{1}{1 + \dfrac{\sin 2m_i^{(2)}\phi}{2m_i^{(2)}\phi}}\int_0^{\phi} \cos m_j^{(1)}\varphi \cos m_i^{(2)}\varphi \,\mathrm{d}\varphi$$

or, after integration

$$\Psi_{i,j} = \frac{1}{1 + \dfrac{\sin 2m_i^{(2)}\phi}{2m_i^{(2)}\phi}}\left[\frac{\sin (m_j^{(1)} + m_i^{(2)})\,\phi}{(m_j^{(1)} + m_i^{(2)})\,\phi} + \frac{\sin (m_j^{(1)} - m_i^{(2)})\,\phi}{(m_j^{(1)} - m_i^{(2)})\,\phi}\right].$$

3.6. *Boundary conditions at the interface between annular region and annular sector in an "eccentric" arrangement*

A schematic diagram of this arrangement is in Fig. 4. Referring to this figure we can write the boundary conditions as

$$\lambda_1 \frac{\partial t_1}{\partial r}\bigg|_{r=R} = -\,\alpha(\varphi)\,[t_1(R,\varphi) - t_0(\varphi)] \qquad \text{for} \quad \phi < \varphi \leqslant \phi_1 \tag{59}$$

$$\lambda_1 \frac{\partial t_1}{\partial r}\bigg|_{r=R} = -\,\lambda_2\left[\frac{\partial t_2}{\partial r}\bigg|_{r=\bar{r}} \cos(\varphi + \varphi') - \frac{\partial t_2}{\partial \varphi}\frac{1}{\bar{r}}\sin(\varphi + \varphi')\right] \qquad \text{for} \quad 0 \leqslant \varphi \leqslant \phi \tag{60}$$

$$t_1(R,\varphi) = t_2(\bar{r},\varphi') \qquad \text{for} \quad 0 \leqslant \varphi \leqslant \phi. \tag{61}$$

The basic difference between the previous boundary conditions and between the presently investigated boundary condition is that the regions which meet at the given radius have each a different coordinate system. Thus it is necessary at the interface of both regions to have in mind the dependence between these coordinate systems. On the whole we shall proceed in an analogous way as in the Section 3.5. Boundary conditions (59) and (60) can be written in the form

$$\lambda_1 \frac{\partial t_1}{\partial r}\bigg|_{r=R} = -\,\alpha(\varphi)\,[t_1(R,\varphi) - t_0(\varphi)] - \delta(\varphi)\,\lambda_2\left[\frac{\partial t_2}{\partial r}\bigg|_{r=\bar{r}} \cos(\varphi + \varphi') - \frac{\partial t_2}{\partial \varphi}\frac{1}{\bar{r}}\sin(\varphi + \varphi')\right]. \tag{62}$$

This equation is multiplied successively by functions

$$\frac{1}{\phi_1} \qquad \text{for} \quad m_i^{(1)} = 0$$

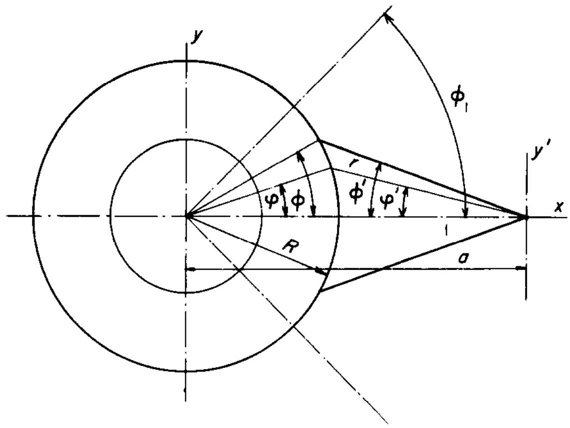


FIG. 4. Annular sector attaching to annular region in eccentric arrangement.

$$\frac{2}{\phi_1} \cos m_i^{(1)}\varphi \qquad \text{for} \quad m_i^{(1)} \neq 0$$

and integrated within the interval of $0 \leqslant \varphi \leqslant \phi_1$, with respect to the independent variable $\varphi$. From geometrical considerations (see Fig. 4) follows that the dependence $\bar{r}$ and $\varphi'$ on the angle $\varphi$ has the form

$$\bar{r} = \sqrt{[(R \sin \varphi)^2 + (a - R \cos \varphi)^2]}$$

$$\sin \varphi' = \sin \varphi \frac{R}{\bar{r}}.$$

We get equations that can be written in matrix notation as follows

$$\lambda_1[{}^1_2\mathbf{M}_R \mathbf{C}_1 + {}^1\mathbf{Z}'_R] = - \mathbf{K}_\alpha[{}^1_2\mathbf{M}_R \mathbf{C}_1 + {}^1\mathbf{Z}_R + \mathbf{T}_{0R}] - \lambda_2[{}_3\mathbf{X}\mathbf{C}_2 + {}_4\mathbf{X}]. \tag{63}$$

Matrices ${}_3\mathbf{X}$ and ${}_4\mathbf{X}$ are respectively in the form of

$${}_3\mathbf{X} = \xi \begin{bmatrix} I_{11}, & J_{11}, & I_{12}, & J_{12}, & \ldots \\ I_{21}, & J_{21}, & I_{22}, & J_{22}, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}, \quad {}_4\mathbf{X} = \xi \begin{bmatrix} \sum_i K_{1j} \\ \sum_j K_{2j} \\ \vdots \end{bmatrix}$$

where

$$I_{i,j} = \int_0^\phi [\cos m_j^{(2)}\varphi' . \cos(\varphi' + \varphi) + \sin m_j^{(2)}\varphi' . \sin(\varphi' + \varphi)]\, m_j^{(2)}\, \bar{r}^{m_j^{(2)}-1} \cos m_i^{(1)}\varphi \, d\varphi$$

$$J_{i,j} = \int_0^\phi [- \cos m_j^{(2)}\varphi' . \cos(\varphi' + \varphi) + \sin m_j^{(2)}\varphi' . \sin(\varphi' + \varphi)]\, m_j^{(2)}\, \bar{r}^{-m_j^{(2)}-1} \cos m_i^{(1)}\varphi \, d\varphi$$

$$K_{i,j} = \int_0^\phi \left[ \frac{m_j^{(2)}}{\bar{r}} (Z_{m_j^{(2)}}(\bar{r}) + t_{m_j^{(2)}}) \sin m_j^{(2)}\varphi' . \sin(\varphi' + \varphi) + Z'_{m_j^{(2)}}(\bar{r}) \cos m_j^{(2)}\varphi' . \cos(\varphi' + \varphi) \right] \cos m_i^{(1)} \varphi \, d\varphi$$

$$\xi = \begin{cases} \dfrac{1}{\phi_1} & (m_i^{(1)} = 0) \\ \dfrac{2}{\phi_1} & (m_i^{(1)} \neq 0). \end{cases}$$

Quantities $Z_{m_j^{(2)}}(\bar{r})$ and $t_{m_j^{(2)}}$ are the elements of vector ${}^2\mathbf{Z}_{\bar{r}}$ or ${}^2\mathbf{T}_0$ [cf. equations (31a) and (13)].

Now we introduce in condition (61) the series for $t_1$ and $t_2$, multiply the equation, similarly as in Section 3.3 or 3.5 by functions of the form of (45) using the values of $m_i^{(2)}$ and integrate within the interval of $0 \leqslant \varphi' \leqslant \phi'$ according to the variable $\varphi'$. The equations thus obtained may be written in matrix notation as follows [refer also to the equation (58)]:

$$ {}_1\mathbf{Y}\,\mathbf{C}_2 + {}_2\mathbf{Y} = \overline{\boldsymbol{\psi}}(\tfrac{1}{2}\mathbf{M}\,\mathbf{C}_1 + {}^1\mathbf{Z}_R). \tag{64}$$

Matrices ${}_1\mathbf{Y}$ and ${}_2\mathbf{Y}$ are respectively in the form of

$$ {}_1\mathbf{Y} = \xi \begin{bmatrix} U_{11}, & V_{11}, & U_{12}, & V_{12}, & \ldots \\ U_{21}, & V_{21}, & U_{22}, & V_{22}, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}, \quad {}_2\mathbf{Y} = \xi \begin{bmatrix} \sum_j W_{1j} \\ \sum_j W_{2j} \\ \vdots \end{bmatrix}$$

where

$$U_{i,j} = \int_0^{\phi'} \bar{r}^{-m_j^{(2)}} \cos m_j^{(2)}\varphi' \,.\, \cos m_i^{(2)}\varphi' \,\mathrm{d}\varphi'$$

$$V_{i,j} = \int_0^{\phi'} \bar{r}^{-m_j^{(2)}} \cos m_j^{(2)}\varphi' \,.\, \cos m_i^{(2)}\varphi' \,\mathrm{d}\varphi'$$

$$W_{i,j} = \int_0^{\phi'} (Z_{m_j^{(2)}}(\bar{r}) + t_{m_j^{(2)}}) \cos m_j^{(2)}\varphi' \,.\, \cos m_i^{(2)}\varphi' \,\mathrm{d}\varphi'$$

$$\xi = \frac{2}{\phi'} \frac{1}{1 + \dfrac{\sin 2m_i^{(2)}\phi'}{2m_i^{(2)}\phi'}}$$

and where

$$\bar{r} = a\cos\varphi' - \sqrt{[R^2 - (a\sin\varphi')^2]}$$

$$\sin\varphi = \frac{\bar{r}}{R}\sin\varphi'.$$

The elements of matrix $\overline{\boldsymbol{\psi}}$ take the form of

$$\overline{\psi}_{i,j} = \xi \int_0^{\phi'} \cos m_j^{(1)}\varphi \,.\, \cos m_i^{(2)}\varphi' \,\mathrm{d}\varphi'.$$

The boundary conditions for the case of a prismatic fin or a fin attached to the annulus from the inside, can be written in an analogous manner.

## 4. BOUNDARY PROBLEMS

Using the results of Section 3 it is possible (for a great number of boundary problems) to compile a resulting infinite system of equations for the calculation of the integration constants. For the actual solution of boundary problems we retain in the infinite series only few first terms of the expansion and solve the remaining finite system of linear algebraic equations.

As already stated the method can be applied to the treatment of many other boundary conditions. In view of the fact that in the analysis of laminar flow or the analysis of heat transfer in laminar flow the partial differential equations which have to be solved are the same as the heat-conduction equation discussed in the foregoing, the method evolved in the paper can be used for solving many a problem of laminar flow in non-circular channels.

We shall now show solutions of some typical boundary problems.

*Example* 1: *Solution of the boundary problem for an annular sector.* Section 3.3 has outlined the procedure to be adopted in the case where the heat-transfer coefficient varies along the periphery of the annular sector (assuming the problem to be symmetrical with respect to the $x$-axis). Consider the annular sector to be subdivided into sub-regions as shown in Fig. 2. For each sub-region on radii $R_k(R_1, R_2, \ldots, R_{k-1})$ we can write the boundary conditions in the form of (46) and (47). For radii $R_0$ and $R_k$, the resultant equations for the boundary conditions are (51) and (50) depending on whether or not $\alpha$ is constant on that segment.

The resultant system of equations can be written as

$$\mathbf{MC} = \mathbf{F} \tag{65}$$

where

$$\mathbf{M} = \begin{bmatrix} \alpha_0\, {}^1_3\mathbf{M}_{R_0} - \lambda\, {}^1_3\mathbf{M}'_{R_0}, & 0, & 0, & \ldots, & 0, & 0, \\ \phi^{2,1}\, {}^1_3\mathbf{M}_{R_1}, & -\, {}^2_3\mathbf{M}_{R_1}, & 0, & \ldots, & 0, & 0 \\ {}^1_3\mathbf{M}'_{R_1}, & -\,\phi^{1,2}\, {}^2_3\mathbf{M}'_{R_1}, & 0, & \ldots, & 0, & 0 \\ 0, & \phi^{3,2}\, {}^2_3\mathbf{M}_{R_2}, & -\, {}^3_3\mathbf{M}_{R_2}, & \ldots, & 0, & 0 \\ 0, & {}^2_3\mathbf{M}_{R_2} & -\,\phi^{2,3}\, {}^3_3\mathbf{M}'_{R_2}, & \ldots, & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & 0, & \ldots, & 0, & \alpha_k\, {}^k_3\mathbf{M}_{R_k} + \lambda\, {}^k_3\mathbf{M}'_{R_k} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \\ \vdots \\ \mathbf{C}_k \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \lambda\, {}^1\mathbf{Z}'_{R_0} - \alpha_0({}^1\mathbf{Z}_{R_0} + {}^1\mathbf{T}_0 - \mathbf{T}_{0R_0}) \\ {}^2\mathbf{Z}_{R_1} + {}^2\mathbf{T}_0 - \phi^{2,1}({}^1\mathbf{Z}_{R_1} + {}^1\mathbf{T}_0) \\ \phi^{1,2}\, {}^2\mathbf{Z}'_{R_1} - {}^1\mathbf{Z}'_{R_1} \\ {}^3\mathbf{Z}_{R_2} + {}^3\mathbf{T}_0 - \phi^{3,2}({}^2\mathbf{Z}_{R_2} + {}^2\mathbf{T}_0) \\ \phi^{2,3}\, {}^3\mathbf{Z}'_{R_2} - {}^2\mathbf{Z}'_{R_2} \\ \cdots \\ -\,\lambda\, {}^k\mathbf{Z}'_{R_k} - \alpha_k({}^k\mathbf{Z}_{R_k} + {}^k\mathbf{T}_0 - \mathbf{T}_{0R_k}) \end{bmatrix}$$

The solution of the system of equations (64) gives us directly all the integration constants, and, when introducing the series for $t(r, \varphi)$, the distribution of temperatures. For the purpose of illustration, Fig. 5 plots the temperature field for the case when $\alpha$ is constant along the whole periphery and $t_0 = 0$. The annular sector was subdivided respectively in two, four and eight subregions. The results of computation for four and eight subregions did not differ from each other. Fig. 5

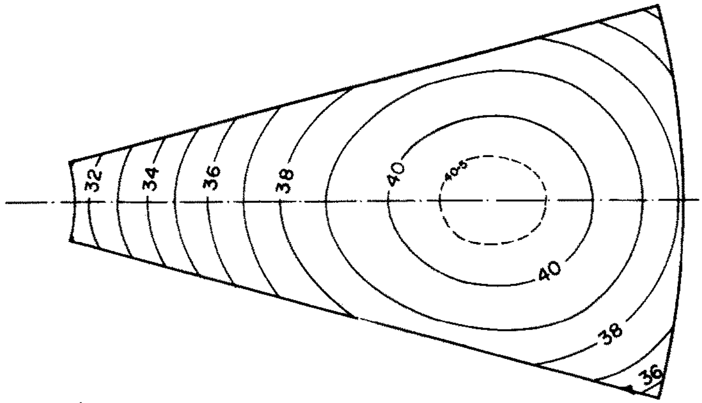


FIG. 5. Temperature field in annular sector in case $\alpha$ is constant ($t_0 = 0$) along periphery.

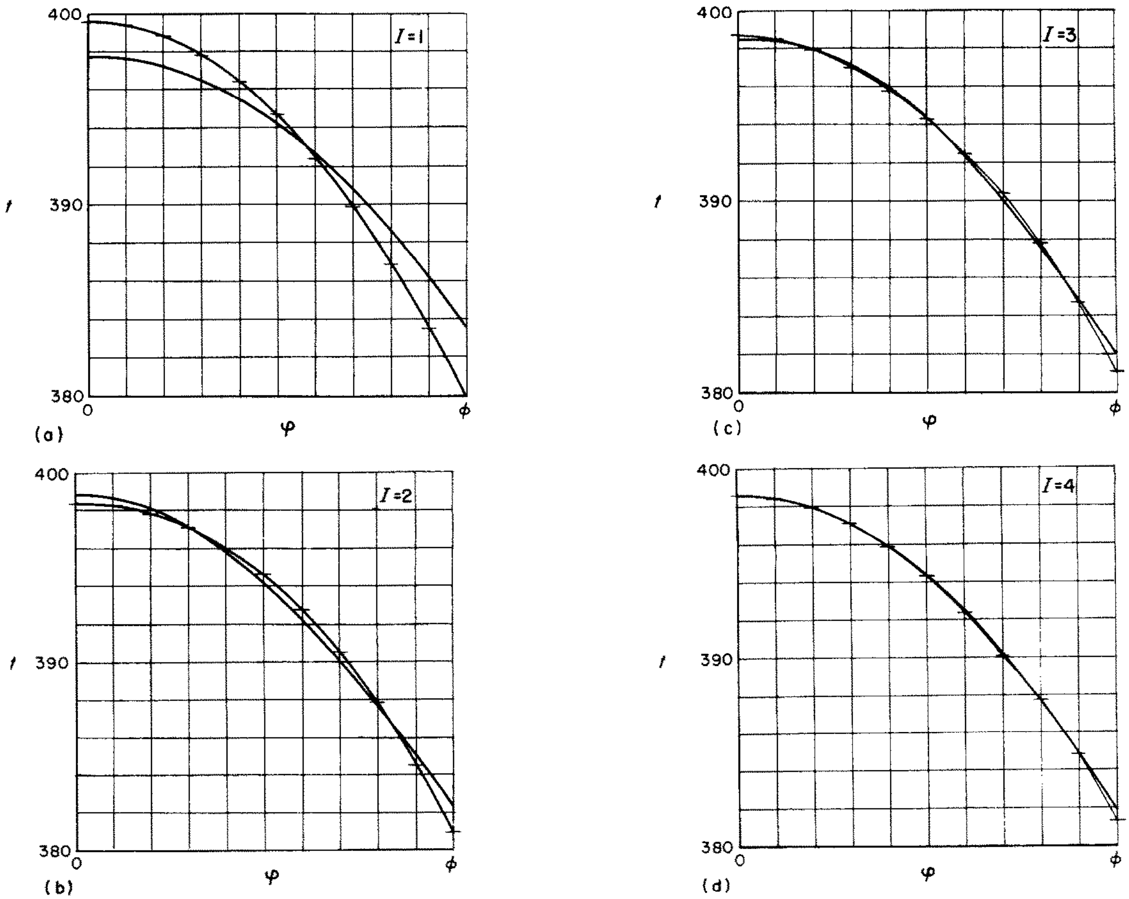


FIG. 6. Fulfilment of the boundary condition (35) on interface of two sub-regions of annular sector for increasing number of expansion terms (I—number of expansion terms).

gives the temperature field for four subregions and for four terms of the expansion in the individual subregions of the annular sector. Fig. 6 indicates in another case how the boundary condition (43) is satisfied at radius $R_k$ for an increasing number of terms in the expansion for $t(r, \varphi)$.

*Example 2: Annulus attached to an annular sector in a concentric arrangement.* The pertinent boundary problem is shown schematically in Fig. 3. The third boundary condition is assumed to apply at radii $R_0$ and $R_2$ [cf. Section 3.1, relation (38), Section 3.4, relation (51)]. The boundary conditions at the interface between annulus and annular sector are described in Section 3.5 by the equations (56) and (58).

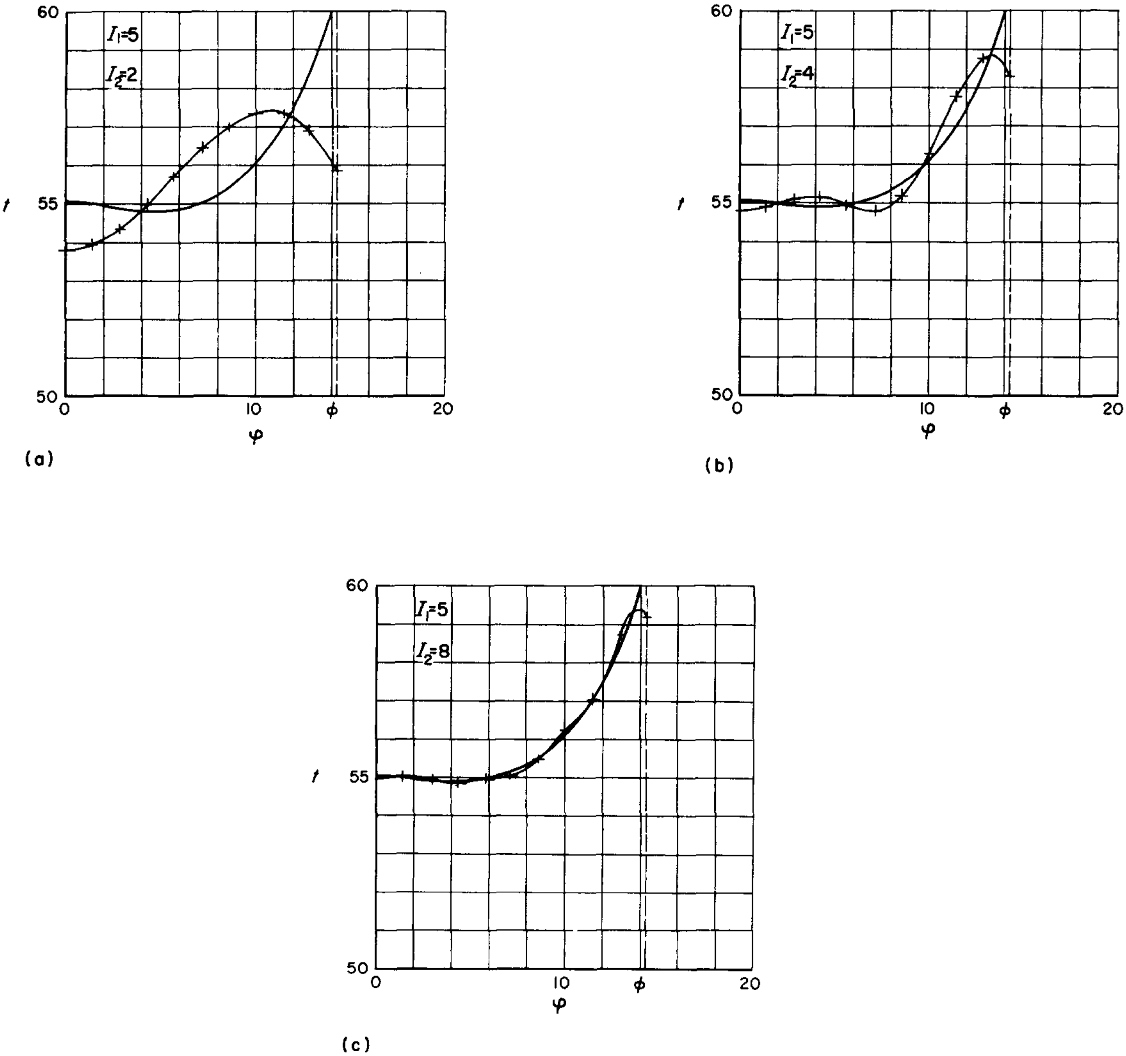


FIG. 7. Fulfilment of the boundary condition (54) on interface between annular sector and annulus for different number of expansion terms in individual regions ($I_1$—number of expansion terms in annular region, $I_2$—number of expansion terms in annular sector).

Joing the equations we obtain

$$\mathbf{MC} = \mathbf{F}$$

where

$$\mathbf{M} = \begin{bmatrix} \alpha_0\, {}^1_2\mathbf{M}_{R_0} - \lambda_1\, {}^1_2\mathbf{M}'_{R_0}, & 0 \\ \mathbf{K}_{\alpha_1}\, {}^1_2\mathbf{M}_{R_1} + \lambda_1\, {}^1_2\mathbf{M}'_{R_1}, & -\lambda_2\, \boldsymbol{\Theta}\, {}^2_3\mathbf{M}'_{R_1} \\ -\boldsymbol{\psi}\, {}^1_2\mathbf{M}_{R_1} & {}^2_3\mathbf{M}_{R_1} \\ \mathbf{0}, & \alpha_2\, {}^2_3\mathbf{M}_{R_2} + \lambda_2\, {}^2_3\mathbf{M}'_{R_2} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} \lambda_1\, {}^1\mathbf{Z}'_{R_0} + \alpha_0(\mathbf{T}_{0R_0} - {}^1\mathbf{Z}_{R_0}) \\ -\lambda_1\, {}^1\mathbf{Z}'_{R_1} - \mathbf{K}_{\alpha_1}({}^1\mathbf{Z}_{R_1} + \mathbf{T}_{0R_1}) + \lambda_2\, \boldsymbol{\Theta}\, {}^2\mathbf{Z}'_{R_1} \\ -{}^2\mathbf{Z}_{R_1} - {}^2\mathbf{T}_0 + \boldsymbol{\psi}\, {}^1\mathbf{Z}_{R_1} \\ -\lambda_2\, {}^2\mathbf{Z}'_{R_2} - \alpha_2({}^2\mathbf{Z}_{R_2} + {}^2\mathbf{T}_0 - \mathbf{T}_{0R_2}) \end{bmatrix}$$

The solution of the system of equations gives the integration constants $\mathbf{C}_1$ and $\mathbf{C}_2$. For the purpose of illustration, Fig. 7 shows by way of a similar case, how the boundary condition (54) is satisfied for different numbers of terms of the expansion in the annulus and fin.

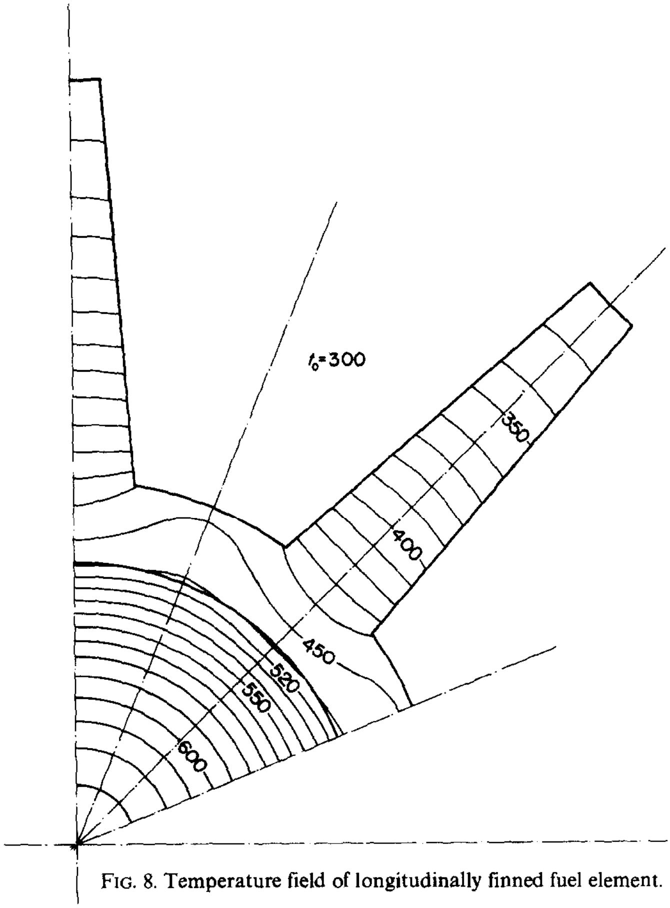


FIG. 8. Temperature field of longitudinally finned fuel element.

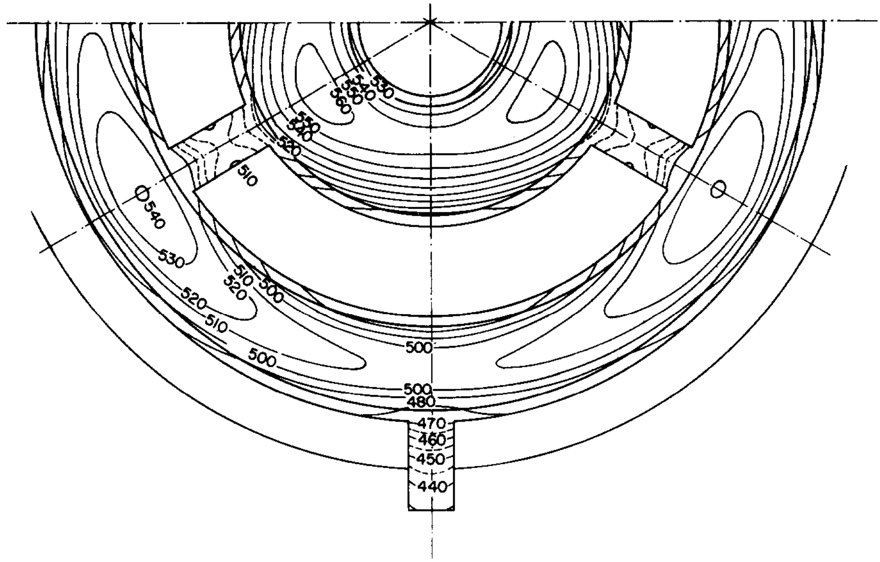


FIG. 9. Temperature field of longitudinally finned fuel element.

*Example* 3. As a practical application of the problems under examination, Figs. 8 and 9 plot the temperature fields of two versions of fuel elements indicated schematically in Fig. 1. The course of the heat-transfer coefficient along the surface of the investigated regions was obtained through the solution of laminar flow in the respective channels, made under some simplifying assumptions. The method used for that purpose was the same as that applied to the solution of the temperature fields.

## 5. CONCLUSION

The method presented in the paper enables us to solve a number of boundary problems of heat conduction in which a cylindrical region is attached to a longitudinal fin. It can equally well be applied to numerous problems of laminar flow in non-circular channels.

## ACKNOWLEDGEMENT

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## REFERENCES

1. F. M. MORS and G. FESHBACH, *Methods of Theoretical Physics* McGraw-Hill, New York (1953).
2. J. SCHMID, Some theoretical problems of heat conduction in reactor fuel elements, Third United Nations International Conference on the Peaceful Uses of Atomic Energy, A/Conf. 28/P/527 (May 1964).

**Résumé**—L'article traite d'une solution du champ de température dans des corps cylindriques consistant en plusieurs cylindres circulaires creux munis d'ailettes longitudinales extérieures ou intérieures ayant la forme de secteurs annulaires ou de prismes. Le problème est analysé avec des conditions aux limites variables le long de la périphérie, et pour une distribution arbitraire de sources sur toute l'étendue de la section droite. Des problèmes de ce type sont rencontrés dans des calculs thermiques d'éléments de combustible de réacteur nucléaire à ailettes longitudinales ou de tubes d'échangeurs de chaleur à ailettes longitudinales. La méthode développée ici peut aussi être appliquée aux problèmes concernant l'écoulement laminaire dans des conduites non-circulaires.

**Zusammenfassung**—Die Arbeit behandelt eine Lösung des Temperaturfeldes in zylindrischen Körpern, die aus verschiedenen Ringräumen bestehen und mit äusseren oder inneren Längsrippen in Form von Ringsektoren oder Prismen versehen sind. Das Problem wird analysiert für Randbedingungen, die sich über den Umfang ändern und für beliebige Quellverteilungen über den Querschnitt. Probleme dieser Art treten auf bei der thermischen Berechnung längsberippter Brennstoffelemente von Kernreaktoren oder längsberippter Rohre von Wärmeübertragern. Die hier entwickelte Methode kann auch auf Probleme der Laminarströmung in nichtkreisförmigen Kanälen angewandt werden.

**Аннотация**—В статье рассматривается решение для температурного поля в цилиндрических телах, состоящих из нескольких колец с внешними или внутренними продольными ребрами в виде секторов кольца или призм. Анализируется задача для граничных условий, меняющихся по окружности и для источников, произвольно распределенных по сечению. Задачи такого типа встречаются в теплотехнических расчетах топливных элементов ядерных реакторов с продольными ребрами или труб теплообменников с продольным оребрением. Метод, использованный в статье, можно также применить к задачам о ламинарном течении в некруглых трубах.